ISRAEL JOURNAL OF MATHEMATICS **124** (2001), 285–297

ON RESTRICTED WEAK TYPE (1,1); THE DISCRETE CASE

BY

MUSTAFA AKCOGLU*

Department of Mathematics, University of Toronto
Toronto, Ontario M5S 1A1, Canada
e-mail: akcoglu@math.toronto.edu

AND

JOHN BAXTER

School of Mathematics, University of Minneapolis
Minneapolis, MN 55455, USA
e-mail: baxter@math.umn.edu

AND

ALEXANDRA BELLOW

Department of Mathematics, Northwestern University
Evanston, IL 60201, USA
e-mail: a_bellow@math.nwu.edu

AND

ROGER L. JONES**

Department of Mathematics, DePaul University
2219 N. Kenmore, Chicago, IL 60614, USA
e-mail: rjones@condor.depaul.edu

ABSTRACT

In this paper we study the relationship between restricted weak type (1,1) and weak type (1,1) for convolution operators on $\mathbb{Z}$.

* M. Akcoglu is partially supported by an NSERC Grant.
** R. Jones is partially supported by NSF Grant DMS-9531526.
Received December 23, 1999

1. Introduction

We consider $\ell^1(\mathbb{Z})$, where $\mathbb{Z}$ denotes the space of integers equipped with the counting measure. For any set $\mathcal{C}$ of nonnegative functions in $\ell^1(\mathbb{Z})$ we define the maximal operator $\mathcal{M}_\mathcal{C}$ on nonnegative $\ell^1(\mathbb{Z})$-functions by

$$\mathcal{M}_\mathcal{C} f = \sup_{\varphi \in \mathcal{C}} \varphi * f. \tag{1.1}$$

The function $\mathcal{M}_\mathcal{C} f$ takes values in $[0, \infty]$ and may not be in $\ell^1(\mathbb{Z})$.

For any subset A of $\mathbb{Z}^d$ we denote the number of points in A by $|A|$. (We will also use $|A|$ to denote the Lebesgue measure of the set A if A is being considered as a subset of $\mathbb{R}^d$. It will be clear from the context if we are considering A as a subset of $\mathbb{Z}^d$ or of $\mathbb{R}^d$.)

For an operator M, if it happens that for some finite constant C we have

$$|\{Mf > \alpha\}| \le \frac{C\|f\|_1}{\alpha} \tag{1.2}$$

for all $\alpha > 0$ and all nonnegative $f \in \ell^1(\mathbb{Z})$, then we say that M is **weak type** (1,1). If for some finite constant C

$$|\{M\mathbf{1}_E > \alpha\}| \le \frac{C|E|}{\alpha} = \frac{C\|\mathbf{1}_E\|_1}{\alpha} \tag{1.3}$$

holds for all $\alpha > 0$ and all finite subsets E of $\mathbb{Z}$, then we will say that M is **restricted weak type** (1,1).

Remark 1.1: The analogous definition holds if we are working on $\ell^1(\mathbb{Z}^d)$ or in $L^1(\mathbb{R}^d)$. For $L^1(\mathbb{R}^d)$ we just need to consider sets $E \subset \mathbb{R}^d$ which have finite Lebesgue measure instead of sets $E \subset \mathbb{Z}^d$ which have finite counting measure.

Clearly the restricted weak type (1,1) property is implied by the ordinary unrestricted weak type (1,1) property. Weak type inequalities are quite important since they are the key to obtaining almost everywhere convergence results. However, they are often very difficult to prove since they fail to be subadditive, that is, the inequality $|\{|f+g| > \lambda\}| \le |\{|f| > \lambda\}| + |\{|g| > \lambda\}|$ is false in general.

Remark 1.2: We do have the inequality $|\{|f + g| > 2\lambda\}| \le |\{|f| > \lambda\}| + |\{|g| > \lambda\}|$, and this is enough to handle the case of only a finite number of operators. If we have a countably infinite family of operators, the lack of a subadditive property causes a serious problem. (However, see E. M. Stein and N. J. Weiss [5] for a positive result in this regard.)

Restricted inequalities have had a long history; see, for example, the article by E. M. Stein and G. Weiss [4] which contains a number of applications and examples. In 1974, K. H. Moon [3] considered convolution operators on $\mathbb{R}^d$, with Lebesgue measure. He proved the following theorem.

THEOREM 1.3 (Moon): *Let K_n, $n = 1, 2, \ldots$, be linear operators in $L^1(\mathbb{R}^d)$, each of the form $K_n f(x) = f \star g_n$ for some $g_n \in L^1(\mathbb{R}^d)$. Let $Mf(x) = \sup_n |K_n f(x)|$. Then M is of restricted weak type $(1,1)$ if and only if M is of weak type $(1,1)$.*

Let δ_a denote the Dirac delta function at the point a, that is, δ_a denotes the unit point mass at the point a. If $K \in L^1(\mathbb{R}^d)$, and $f = \sum_{i=1}^n \delta_{a_i}$, where $\{a_1, a_2, \ldots, a_n\}$ is a set of distinct points in $\mathbb{R}^d$, then

$$Kf(x) = K \star \Big(\sum_{i=1}^{n} \delta_{a_i}\Big)(x) = \sum_{i=1}^{n} K(x - a_i).$$

In [2] de Guzmán proved the following related result.

THEOREM 1.4: *Let K_n, $n = 1, 2, \ldots$, be linear operators in $L^1(\mathbb{R}^d)$, each of the form $K_n f(x) = f \star g_n$ for some $g_n \in L^1(\mathbb{R}^d)$. Let $Mf(x) = \sup_n |K_n f(x)|$. Then M is weak type $(1,1)$ if and only if there is a constant c such that for all $H > 0$, and distinct (a_h), we have*

$$\Big|\Big\{x\colon M\Big(\sum_{h=1}^{H} \delta_{a_h}\Big)(x) > \lambda\Big\}\Big| = \Big|\Big\{x : \sup_n \Big|\sum_{h=1}^{H} g_n(x - a_h)\Big| > \lambda\Big\}\Big| \le c\frac{H}{\lambda}.$$

Remark 1.5: It is natural to ask the following question: Does de Guzmán's conclusion follow if we only know $|\{x\colon M\delta_a(x) > \lambda\}| \le c/\lambda$? That is, can the finite sum of Dirac delta functions be replaced by a single Dirac delta function? To see that the answer to this is no, we argue as follows. We know that on $\mathbb{Z}$ we do not have a weak type (1,1) inequality for the family of operators $A_n\phi(s) = \frac{1}{n}\sum_{k=1}^n \phi(s + 2^k)$. (See [1] for this and other related examples, as well as much stronger statements.) Writing $K_n f(x) = \frac{1}{n}\sum_{k=1}^n f \star \chi_{[2^k, 2^k+1)}(x)$, and using the result on $\mathbb{Z}$ from [1], we can see that the maximal operator is not a weak type (1,1) operator. However, it is not hard to show that we do have the required inequality when we apply the maximal operator to functions of the form δ_a.

Given the results of Moon and de Guzmán, it is natural to expect that the same result would hold if $\mathbb{R}^d$ is replaced by $\mathbb{Z}^d$. The proof of both Moon's result and de Guzmán's result takes advantage of the nonatomic structure of Lebesgue measure. Thus it was clear that to obtain the $\mathbb{Z}^d$ version of their results, a new proof would be required. Unfortunately, all our attempts to prove such a result led to unexpected problems, and we began to wonder if the $\mathbb{Z}^d$ analogs of their results might be false. In this paper we construct an example to show that indeed the discrete analog of their results is false.

As a convenient notation we will let $c_u = c_u(M)$ denote the smallest constant C which makes (1.2) true for all $\alpha > 0$ and all nonnegative $f \in \ell^1(\mathbb{Z})$, with the convention that $c_u \equiv \infty$ if M is not weak type (1,1). Similarly, we will let $c_r = c_r(M)$ denote the smallest constant C which makes (1.3) true for all $\alpha > 0$ and all finite subsets E of $\mathbb{Z}$, again letting $c_r \equiv \infty$ if M is not restricted weak type (1,1). Naturally we have $c_r \leq c_u$, but it is not immediately clear what more can be said.

We will show that it is possible to construct examples such that $c_r < \infty$ while $c_u = \infty$. Our examples will have an additional property, namely that the convolution kernels $\varphi \in \mathcal{C}$ will be nonnegative and normalized to have integral one with respect to counting measure. Such functions are often referred to as probability densities. Thus we will prove the following.

THEOREM 1.6: *There exists a countable set $\mathcal{C}$ of probability densities such that $\mathcal{M}_{\mathcal{C}}$ is restricted weak type (1,1) but not weak type (1,1).*

2. Some reductions

In this section we show that we can reduce the problem to finding measures such that the ratio c_u/c_r is large.

LEMMA 2.1: *Let γ be a nonnegative real number. Suppose that for every positive real number K, there exists a finite set $\mathcal{C}$ of probability densities, such that $c_r(\mathcal{M}_{\mathcal{C}}) \leq \gamma$ and $c_u(\mathcal{M}_{\mathcal{C}}) \geq K$. Then Theorem 1.6 holds.*

Proof: First note that if we can find a collection $\mathcal{C}$ of probability densities for which the maximal function is restricted weak type (1,1), but fails to be weak type (1,1), then we can find a countable subcollection of $\mathcal{C}$ with the same property. To see this, assume we have a collection $\mathcal{C}$ such that the maximal operator $\mathcal{M}_{\mathcal{C}}$ fails to be weak type (1,1). Then for each n we can find a $\lambda_n > 0$, a function f_n, and a sequence $\phi_{n,k}$ such that

$$|\{\sup_k |\phi_{n,k} \star f_n| > \lambda_n\}| > \frac{n}{\lambda_n}\|f_n\|_1.$$

Clearly the collection $\{\phi_{n,k} : n = 1, 2, \ldots,\ k = 1, 2, \ldots\}$ is a countable collection for which the maximal operator fails to be weak type (1,1). On the other hand, since it is a subcollection of $\mathcal{C}$, the maximal operator must still be restricted weak type (1,1).

For each $j = 1, 2, \ldots$, let $\mathcal{C}_j$ be a finite set of probability densities such that, if $M_j \equiv \mathcal{M}_{\mathcal{C}_j}$, then $c_r(M_j) \leq \gamma$ and $c_u(M_j) > j4^j$. Then for each j, there is a nonnegative function $f_j \in \ell^1(\mathbb{Z})$ and some $\alpha_j > 0$ such that

$$|\{M_j f_j > \alpha_j\}| > \frac{j4^j}{\alpha_j}\|f_j\|_1.$$

Define $\mathcal{D}$ to be the set of all functions φ of the form

$$\varphi = 3\sum_{j=1}^{\infty} \frac{1}{4^j}\varphi_j,$$

where $\varphi_j \in \mathcal{C}_j$. We have

$$\mathcal{M}_{\mathcal{D}} = 3\sum_{j=1}^{\infty} \frac{1}{4^j} M_j.$$

For every finite subset E of $\mathbb{Z}$ and every $\alpha > 0$ we have

$$\begin{aligned}
|\{\mathcal{M}_{\mathcal{D}}\mathbf{1}_E > \alpha\}| &\leq \left|\left\{3\sum_{j=1}^{\infty}\frac{1}{4^j}M_j\mathbf{1}_E > \alpha\right\}\right| \\
&\leq \left|\bigcup_{j=1}^{\infty}\left\{\frac{3}{4^j}M_j\mathbf{1}_E > \frac{\alpha}{2^j}\right\}\right| \\
&\leq \sum_{j=1}^{\infty}\left|\left\{M_j\mathbf{1}_E > \frac{2^j\alpha}{3}\right\}\right| \\
&\leq \sum_{j=1}^{\infty}\frac{3\gamma}{2^j\alpha}|E| = \frac{3\gamma}{\alpha}|E|.
\end{aligned}$$

Thus $\mathcal{M}_{\mathcal{D}}$ is restricted weak type (1,1).

Also

$$\begin{aligned}
\left|\left\{Mf_j > \frac{3\alpha_j}{4^j}\right\}\right| &\geq \left|\left\{\frac{3}{4^j}M_jf_j > \frac{3\alpha_j}{4^j}\right\}\right| \\
&= |\{M_jf_j > \alpha_j\}| \\
&> \frac{j4^j}{\alpha_j}\|f_j\|_1 \\
&= \frac{3j}{3\alpha_j/4^j}\|f_j\|_1.
\end{aligned}$$

It follows that $\mathcal{M}_{\mathcal{D}}$ is not weak type (1,1), and the lemma is proved. ∎

LEMMA 2.2: *Suppose that for every real number ε, with $0 < \varepsilon < 1$, there exists a finite set $\mathcal{W}$ of nonnegative convolution kernels in $\ell^1(\mathbb{Z})$ such that for all $\varphi \in \mathcal{W}$*

$$\|\varphi\|_1 \leq \varepsilon, \tag{2.1}$$

$$c_r(\mathcal{M}_{\mathcal{W}}) \leq \varepsilon, \tag{2.1}$$

and

$$c_u(\mathcal{M}_{\mathcal{W}}) \geq 1. \tag{2.1}$$

Then Theorem 1.6 holds.

Proof: Let a positive real number ε be given. Let $K = 1/\varepsilon$, and let $\mathcal{W}$ satisfy the hypotheses of the present lemma, so that (2.1), (2.2) and (2.3) hold.

For any $\varphi \in \mathcal{W}$, first enlarge φ by multiplication by K. The nonnegative density $K\varphi$ satisfies $\|K\varphi\|_1 \leq 1$. To create a density with norm 1, let

$$b(\varphi) = 1 - K\|\varphi\|_1.$$

By assumption, $0 \leq b(\varphi) \leq 1$. Let

$$\mathcal{C} = \{K\varphi + b(\varphi)\delta_0 \colon \varphi \in \mathcal{W}\},$$

where, as usual, we denote the function which is 1 at the point a and zero elsewhere by δ_a. Now, by the construction, $\mathcal{C}$ is a set of probability densities. Clearly

$$\mathcal{M}_{\mathcal{C}} \leq K\mathcal{M}_{\mathcal{W}} + I,$$

where I denotes the identity operator. Then for any $\alpha > 0$ and any finite subset E of $\mathbb{Z}$ we have

$$\begin{aligned} |\{\mathcal{M}_{\mathcal{C}}\mathbf{1}_E > \alpha\}| &\leq \left|\left\{K\mathcal{M}_{\mathcal{W}}\mathbf{1}_E > \frac{\alpha}{2}\right\}\right| + \left|\left\{\mathbf{1}_E > \frac{\alpha}{2}\right\}\right| \\ &\leq \frac{2|E|}{\alpha} + \frac{2|E|}{\alpha}. \end{aligned}$$

Thus $c_r(\mathcal{M}_{\mathcal{C}}) \leq 4$. On the other hand, clearly $c_u(\mathcal{M}_{\mathcal{C}}) \geq K$. We can now apply Lemma 2.1 to conclude that Theorem 1.6 holds, as claimed. ∎

3. The construction

We will construct a set $\mathcal{W}$ which satisfies the hypotheses of Lemma 2.2. We can then apply Lemma 2.2 to complete the proof of Theorem 1.6.

3.1. DEFINITION OF $\mathcal{W}$. For our construction, we start with a divergent series $a_0 + a_1 + a_2 + \cdots$ with $1 \geq a_0 \geq a_1 \geq \cdots > 0$, and satisfying

$$\sup_{0 \leq j < n} \frac{(j+1)a_j}{\sum_{k=0}^{n-1} a_k} \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Let

$$\zeta_n = \frac{1}{\sum_{k=0}^{n-1} a_k},$$

and note that by assumption $\zeta_n \to 0$ as $n \to \infty$.

For each n we now define a very nonuniform probability density $\lambda(\cdot, n)$ on $\mathbb{Z}$ as follows:

$$\lambda(j,n) = \begin{cases} \zeta_n\, a_j & \text{for } j = 0, 1, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then condition (3.1) can be rewritten as

$$\gamma_n = \sup_{0 \leq j < n} (j+1)\lambda(j,n) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Remark 3.1: Examples of series that satisfy the above conditions include $a_j = 1/(j+1)$, $j = 0, 1, \ldots$ and $a_j = 1/(j+1)\log(j+3)$. However, it is easier to just think about the properties mentioned above.

Let $\varepsilon > 0$ be given and fix m such that $\zeta_m < \varepsilon$ and $\gamma_m < \varepsilon/4$. Such an m exists because of condition (3.2) and the fact that $\zeta_n \to 0$. For this fixed m, and each integer j, let $\lambda(j) = \lambda(j,m)$. Let $\beta \geq m+1$ be an integer which also satisfies $\beta - 1 \geq 1/a_m$.

Let

$$H = \{0, 1, \ldots, m\beta^m - 1\},$$

so that H is a long "interval" containing $m\beta^m$ points. All the interesting action of our construction will take place on H.

We will need shifted versions of λ, which are defined as follows. For each $\ell \in \{0, 1, \ldots, m-1\}$, and each $y \in \mathbb{Z}$, let

$$\lambda_\ell(y) \equiv \lambda(i),$$

where $0 \leq i \leq m-1$ and $i \equiv y + \ell \bmod m$. Thus $\lambda_\ell(y) = \lambda(y \oplus \ell)$ where $\oplus$ denotes addition mod m. The function λ_ℓ is thus periodic, and the sum of the values of λ_ℓ over any period is 1. Furthermore,

$$\sum_{\ell=0}^{m-1} \lambda_\ell(y) \equiv 1 \quad \text{for each } y \in \mathbb{Z}.$$

For each $\ell \in \{0,1,\ldots,m-1\}$, we define

$$P_\ell = \{t\, m\beta^\ell + y\colon t \in \{0,1,\ldots,\beta^{m-\ell}-1\},\ y \in \{0,1,\ldots,m-1\}\}.$$

Thus P_ℓ consists of $\beta^{m-\ell}$ blocks, each of length m, and is periodic in H with period $m\beta^\ell$. Define

$$\mathcal{W}_\ell = \{\lambda_\ell(x)\delta_x\colon x \in P_\ell\}.$$

Let

$$\mathcal{W} = \left\{ \sum_{\ell=0}^{m-1} \frac{1}{\beta^{m-\ell}}\varphi_\ell\colon \varphi_\ell \in \mathcal{W}_\ell \text{ for } \ell = 0,1,\ldots,m-1 \right\}.$$

Let $\mathcal{M}_\mathcal{W}$ denote the maximal operator obtained by convolution with densities in $\mathcal{W}$, and for each $\ell = 0,1,\ldots,m-1$, let $M_\ell = \mathcal{M}_{\mathcal{W}_\ell}$ denote the maximal operator obtained by convolution with densities in $\mathcal{W}_\ell$.

Clearly

$$\mathcal{M}_\mathcal{W} = \sum_{\ell=0}^{m-1} \frac{1}{\beta^{m-\ell}} M_\ell.$$

3.2. Proof of condition (2.1). For any $\varphi_\ell \in \mathcal{W}_\ell$, we have $\|\varphi_\ell\|_1 \le \zeta_m$. Hence for any $\varphi \in \mathcal{W}$ we have

$$\|\varphi\|_1 \le \sum_{\ell=0}^{m-1} \frac{1}{\beta^{m-\ell}}\zeta_m < \frac{\zeta_m}{\beta-1} < \zeta_m < \epsilon.$$

Consequently condition (2.1) of Lemma 2.2 holds.

3.3. Proof of condition (2.3). We will define a particular function f which is a superposition of spikes of widely varying heights. The spikes are nevertheless located in such a way that Mf is large on a substantial portion of the space.

For each $i = 0,1,\ldots,m-1$, let

$$Q_i = \{0, m, 2m, \ldots, (\beta^i - 1)m\},$$

and let $f_i = \beta^{m-i}\mathbf{1}_{Q_i}$. Thus $\|f_i\|_1 = \beta^m$. Note that, as i decreases, the height of f_i increases and the size of the support of f_i decreases. Let

$$f = \sum_{i=0}^{m-1} f_i.$$

Since each f_i is nonnegative, we have $\|f\|_1 = \sum_{i=0}^{m-1} \|f_i\|_1 = m\beta^m$.

For any two sets A and B, of integers, we use the notation

$$A + B = \{x + y: x \in A,\ y \in B\}.$$

We note that for each $\ell = 0, 1, \ldots, m-1$ we have

$$P_\ell + Q_\ell = H,$$

where H is the long interval defined above.

Now we have to estimate $\mathcal{M}_\mathcal{W} f$. We begin by estimating $M_\ell f_\ell$. Fix $a \in H$. Since $P_\ell + Q_\ell = H$, we know there exists $x \in P_\ell$ and $y \in Q_\ell$ such that $a = x + y$. Hence

$$M_\ell \mathbf{1}_{Q_\ell}(a) \geq \lambda_\ell(x)\delta_x * \delta_y(a) = \lambda_\ell(x)\delta_a(a).$$

However, if $y \in Q_\ell$ then $y \equiv 0 \bmod m$ and thus $a = x + y \equiv x \bmod m$. Since λ_ℓ is periodic with period m, we have $\lambda_\ell(x) = \lambda_\ell(a)$. It follows that

$$M_\ell \mathbf{1}_{Q_\ell}(a) \geq \lambda_\ell(a)\mathbf{1}_H(a),$$

and so

$$M_\ell f_\ell(a) \geq \beta^{m-\ell}\lambda_\ell(a)\mathbf{1}_H(a).$$

Hence

$$\mathcal{M}_\mathcal{W} f(a) \geq \sum_{\ell=0}^{m-1} \frac{1}{\beta^{m-\ell}} M_\ell f_\ell(a) \geq \sum_{\ell=0}^{m-1} \lambda_\ell(a)\mathbf{1}_H(a) = \mathbf{1}_H(a).$$

To have a weak (1,1) inequality (for $\alpha = 1$) we would need to have

$$|\{a: \mathcal{M}_\mathcal{W} f(a) \geq 1\}| \leq \frac{c_u(\mathcal{M}_\mathcal{W})}{1}\|f\|_1,$$

which implies we must have the inequality

$$m\beta^m \leq \frac{c_u(\mathcal{M}_\mathcal{W})}{1}(m\beta^m).$$

Thus we have $c_u(\mathcal{M}_\mathcal{W}) \geq 1$, and we see that condition (2.3) of Lemma 2.2 is satisfied.

3.4. Proof of condition (2.2). We will need the following simple lemma about a set of numbers.

LEMMA 3.2: *Fix real numbers $0 < a < b$ and let $t_0, t_1, \dots, t_{m-1}$ be nonnegative numbers such that if $t_i > 0$ then $a \le t_i \le b$. Let $\beta > 1$ be large enough that $\beta - 1 \ge b/a > 1$. Assume that*

$$\sum_{k=0}^{m-1} \frac{1}{\beta^{m-k}} t_k > \alpha$$

for some positive real number α. Let ℓ be the maximal index such that $t_\ell > 0$. Then

$$\frac{1}{\beta^{m-\ell}} t_\ell > \frac{\alpha}{2}.$$

Proof: First note that

$$\sum_{i=0}^{\ell-1} \frac{b}{\beta^{m-i}} \le \frac{a}{\beta^{m-\ell}}.$$

This follows since

$$\begin{aligned}\sum_{i=0}^{\ell-1} \frac{b}{\beta^{m-i}} &= \frac{b}{\beta^{m-\ell}} \sum_{i=0}^{\ell-1} \frac{1}{\beta^{\ell-i}} \\ &\le \frac{b}{\beta^{m-\ell}} \frac{1}{\beta - 1} \\ &\le \frac{b}{\beta^{m-\ell}} \frac{1}{b/a} \\ &\le \frac{a}{\beta^{m-\ell}}.\end{aligned}$$

If the lemma were false, then we would have

$$\frac{1}{\beta^{m-\ell}} t_\ell \le \frac{\alpha}{2}.$$

From this we see that

$$\begin{aligned}\sum_{k=0}^{m-1} \frac{1}{\beta^{m-k}} t_k &= \sum_{k=0}^{\ell-1} \frac{1}{\beta^{m-k}} t_k + \frac{1}{\beta^{m-\ell}} t_\ell \\ &\le \sum_{k=0}^{\ell-1} \frac{b}{\beta^{m-k}} + \frac{1}{\beta^{m-\ell}} t_\ell \\ &\le \frac{a}{\beta^{m-\ell}} + \frac{1}{\beta^{m-\ell}} t_\ell \\ &\le \frac{t_\ell}{\beta^{m-\ell}} + \frac{1}{\beta^{m-\ell}} t_\ell \\ &\le \frac{\alpha}{2} + \frac{\alpha}{2} \le \alpha,\end{aligned}$$

a contradiction. ∎

From this we have the following corollary.

COROLLARY 3.3: *Let E be a finite subset of $\mathbb{Z}$. Let $\alpha > 0$ be given. Let $z \in \{\mathcal{M}_{\mathcal{W}}\mathbf{1}_E > \alpha\}$. Then there is some $\ell \in \{0, 1, \ldots, m-1\}$ such that*

$$\frac{1}{\beta^{m-\ell}} M_\ell \mathbf{1}_E(z) > \frac{\alpha}{2}.$$

Proof: We will first establish the hypothesis necessary to apply the previous lemma. Fix $\ell \in \{0, 1, \ldots, m-1\}$. For any point z, by definition there is some $\varphi_\ell \in \mathcal{W}_\ell$ such that $M_\ell \mathbf{1}_E(z) = \varphi_\ell * \mathbf{1}_E(z)$. For some $x \in P_\ell$ we have $\varphi_\ell = \lambda_\ell(x)\delta_x$. Hence, if $M_\ell \mathbf{1}_E(z) > 0$ there is some point $x \in P_\ell$ and some point $v \in E$ such that $x + v = z$, and $M_\ell \mathbf{1}_E(z) = \lambda_\ell(x)$. In particular, we have shown that if $M_\ell \mathbf{1}_E(z) > 0$ then we have $\zeta_m a_m \leq M_\ell \mathbf{1}_E(z) \leq \zeta_m$. Now apply Lemma 3.2. ∎

We now continue with the proof of condition (2.2). Let E be a finite subset of $\mathbb{Z}$ and let $\alpha > 0$ be given. As a consequence of Corollary 3.3, we know that if $z \in \{\mathcal{M}_{\mathcal{W}}\mathbf{1}_E > \alpha\}$ then there is some index $\ell \in \{0, 1, \ldots, m-1\}$, some $x \in P_\ell$ and some $v \in E$ such that the following two conditions hold:

$$z = x + v \tag{3.3}$$

and

$$\frac{\lambda_\ell(x)}{\beta^{m-\ell}} > \frac{\alpha}{2}. \tag{3.4}$$

We will use (3.3) and (3.4) to estimate $|\{\mathcal{M}_{\mathcal{W}}\mathbf{1}_E > \alpha\}|$.

For any $\ell \in \{0, 1, \ldots, m-1\}$, let N_ℓ denote the number of points $x \in P_\ell$ which satisfy (3.4). Let G_ℓ denote the number of points $y \in \{0, 1, \ldots, m-1\}$ which satisfy (3.4). Clearly $N_\ell = \beta^{m-\ell} G_\ell$. Then

$$|\{\mathcal{M}_{\mathcal{W}}\mathbf{1}_E > \alpha\}| \leq |E| \sum_{\ell=0}^{m-1} N_\ell = |E| \sum_{\ell=0}^{m-1} \beta^{m-\ell} G_\ell.$$

We must estimate G_ℓ. By definition,

$$G_\ell = |\{j \colon j \in \{0, \ldots, m-1\},\ \zeta_m a_j / \beta^{m-\ell} > \alpha/2\}|.$$

Thus

$$G_\ell = |\{j \colon j \in \{0, \ldots, m-1\},\ \lambda(j) > \beta^{m-\ell}\alpha/2\}|.$$

We first show that for each ℓ, $0 \le \ell \le m-1$, we have

$$G_\ell \le \frac{2\gamma_m}{\alpha\beta^{m-\ell}}.$$

If we can find an integer j', $0 \le j' < m-1$, such that

$$\lambda(j') > \beta^{m-\ell}\frac{\alpha}{2} \ge \lambda(j'+1),$$

then we have

$$G_\ell = |\{j\colon \lambda(j) > \beta^{m-\ell}\alpha/2\}| = j'+1.$$

Using the definition of j', we see that

$$\beta^{m-\ell}\frac{\alpha}{2}\left|\left\{j\colon \lambda(j) > \beta^{m-\ell}\frac{\alpha}{2}\right\}\right| \le \lambda(j')(j'+1) \le \sup_{0\le j<m}(j+1)\lambda(j) = \gamma_m.$$

Therefore,

$$G_\ell = \left|\left\{j\colon \lambda(j) > \beta^{m-\ell}\frac{\alpha}{2}\right\}\right| \le \frac{2\gamma_m}{\beta^{m-\ell}\alpha}.$$

There are two cases when we cannot find an integer j' as above. The first case is when $\beta^{m-\ell}\alpha/2 \ge \lambda(0)$. In that case we have

$$G_\ell = |\{j\colon \lambda(j) > \beta^{m-\ell}\alpha/2\}| = 0,$$

and there is nothing to prove.

In the second case, when $\beta^{m-\ell}\alpha/2 < \lambda(m-1)$, we have

$$\beta^{m-\ell}\frac{\alpha}{2}\left|\left\{j\colon \lambda(j) > \beta^{m-\ell}\frac{\alpha}{2}\right\}\right| \le \lambda(m-1)m \le \gamma_m,$$

and hence again

$$G_\ell \le \frac{2\gamma_m}{\alpha\beta^{m-\ell}}.$$

Let k be the smallest value of ℓ such that

$$1 \le \frac{2\gamma_m}{\alpha\beta^{m-\ell}}$$

if there is such an ℓ. Otherwise

$$\frac{2\gamma_m}{\alpha\beta^{m-\ell}} < 1 \quad \text{for all } \ell, \quad 0 \le \ell < m$$

and hence $G_\ell = 0$ for all ℓ.

Thus we see that we either have

$$\sum_{\ell=0}^{m-1} \beta^{m-\ell} G_\ell = 0$$

or else for some k we have

$$\begin{aligned}\sum_{\ell=0}^{m-1} \beta^{m-\ell} G_\ell &= \sum_{\ell=k}^{m-1} \beta^{m-\ell} G_\ell \\ &< \beta^{m-k} \frac{2\gamma_m}{\alpha\beta^{m-k}} + \sum_{\ell=k+1}^{m-1} m\beta^{m-\ell} \\ &< \frac{2\gamma_m}{\alpha} + \frac{m}{\beta-1}\beta^{m-k} \\ &< \frac{2\gamma_m}{\alpha} + \frac{2\gamma_m}{\alpha} \\ &= \frac{4\gamma_m}{\alpha}.\end{aligned}$$

We conclude that

$$|\{\mathcal{M}_{\mathcal{W}} \mathbf{1}_E > \alpha\}| \leq |E| \frac{4\gamma_m}{\alpha},$$

which is the same as

$$|\{z: \mathcal{M}_{\mathcal{W}} \mathbf{1}_E(z) > \alpha\}| \leq \frac{4\gamma_m}{\alpha} \|\mathbf{1}_E\|_1,$$

and hence that

$$c_r(\mathcal{M}_{\mathcal{W}}) \leq 4\gamma_m < \epsilon.$$

Consequently condition (2.2) of Lemma 2.2 holds.

Since all the conditions of Lemma 2.2 hold, the conclusion also holds, and the proof of Theorem 1.6 is complete.

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